

ON ASYMPTOTICS OF q -GAMMA FUNCTIONS

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ABSTRACT. In this paper we derive some asymptotic formulas for the q -Gamma function $\Gamma_q(z)$ for q tending to 1.

1. INTRODUCTION

Given complex numbers

$$(1) \quad 0 < q < 1, \quad a \in \mathbb{C},$$

we define [1, 2, 3, 4]

$$(2) \quad (a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k),$$

and the q -Gamma function

$$(3) \quad \Gamma_q(z) := \frac{(q; q)_\infty}{(q^z; q)_\infty} (1 - q)^{1-z} \quad z \in \mathbb{C}.$$

The Euler Gamma function $\Gamma(z)$ is defined as [1, 2, 3, 4]

$$(4) \quad \frac{1}{\Gamma(z)} = z \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) \left(1 + \frac{1}{k}\right)^{-z}, \quad z \in \mathbb{C}.$$

The Gamma function satisfies the reflection formula

$$(5) \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad z \in \mathbb{C},$$

and the integral representation

$$(6) \quad \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \Re(z) > 0.$$

The Gamma function is a very important function in the theory of special functions, since all the hypergeometric series are defined in terms of the shifted factorials $(a)_n$, which are quotients of two Gamma functions

$$(7) \quad (a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}, \quad a \in \mathbb{C}, \quad n \in \mathbb{Z}.$$

Similarly, the q -Gamma function is also very important in the theory of the basic hypergeometric series, because all the basic hypergeometric series are defined in

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terms of the q -shifted factorials $(a; q)_n$, which are scaled quotients of the q -Gamma functions

$$(8) \quad (q^\alpha; q)_n = \frac{(1-q)^n \Gamma_q(\alpha+n)}{\Gamma_q(\alpha)}, \quad \alpha \in \mathbb{C}, \quad n \in \mathbb{Z}.$$

W. Gosper heuristically argued that [1, 2, 3, 4]

$$(9) \quad \lim_{q \rightarrow 1} \frac{1}{\Gamma_q(z)} = \frac{1}{\Gamma(z)}, \quad z \in \mathbb{C}.$$

For a rigorous proof of the case $z \in \mathbb{R}$, see [1]. In this short note we are going to derive some asymptotic formulas for $\Gamma_q(z)$ as $q \rightarrow 1$ in two different modes. In the first mode we let $z \rightarrow \infty$ and $q \rightarrow 1$ simultaneously, while in the second mode we let $q \rightarrow 1$ for a fixed z .

Lemma 1.1. *Given any complex number a , assume that*

$$(10) \quad 0 < \frac{|a|q^n}{1-q} < \frac{1}{2}$$

for some positive integer n . Then, for any positive integer K , we have

$$(11) \quad \frac{(a; q)_n}{(a; q)_\infty} = \frac{1}{(aq^n; q)_\infty} := \sum_{k=0}^{K-1} \frac{(aq^n)^k}{(q; q)_k} + r_1(a, n, K)$$

with

$$(12) \quad |r_1(a, n, K)| \leq \frac{2(|a|q^n)^K}{(q; q)_K},$$

and

$$(13) \quad \frac{(a; q)_\infty}{(a; q)_n} = (aq^n; q)_\infty := \sum_{k=0}^{K-1} \frac{q^{k(k-1)/2}}{(q; q)_k} (-aq^n)^k + r_2(a, n, K)$$

with

$$(14) \quad |r_2(a, n, K)| \leq \frac{2q^{K(K-1)/2}(|a|q^n)^K}{(q; q)_K}.$$

Proof. From the q -binomial theorem [1, 2, 3, 4]

$$(15) \quad \frac{(az; q)_\infty}{(z; q)_\infty} = \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} z^k \quad a, z \in \mathbb{C},$$

we obtain

$$r_1(a, n, K) = \sum_{k=K}^{\infty} \frac{(aq^n)^k}{(q; q)_k} = \frac{(aq^n)^K}{(q; q)_K} \sum_{k=0}^{\infty} \frac{(aq^n)^k}{(q^{K+1}; q)_k}.$$

Since

$$(q^{K+1}; q)_k \geq (1-q)^k$$

for $k = 0, 1, \dots$, thus,

$$|r_1(a, n, K)| \leq \frac{(|a|q^n)^K}{(q; q)_K} \sum_{k=0}^{\infty} \left(\frac{|a|q^n}{1-q} \right)^k \leq \frac{2(|a|q^n)^K}{(q; q)_K}.$$

Apply a limiting case of (15),

$$(16) \quad (z; q)_\infty = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q; q)_k} (-z)^k \quad z \in \mathbb{C},$$

we get

$$r_2(a, n, K) = \frac{q^{K(K-1)/2} (-aq^n)^K}{(q; q)_K} \sum_{k=0}^{\infty} \frac{(-aq^n)^k q^{k(k+2K-1)/2}}{(q^{K+1}; q)_k}.$$

From the inequalities,

$$\frac{1-q^k}{1-q} \geq kq^{k-1}, \quad \frac{(q^{K+1}; q)_k}{(1-q)^k} \geq k! q^{k(k+2K-1)/2}, \quad \text{for } k = 0, 1, \dots$$

we obtain

$$\begin{aligned} |r_2(a, n, K)| &\leq \frac{q^{K(K-1)/2} (|a|q^n)^K}{(q; q)_K} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{|a|q^n}{1-q} \right)^k \\ &\leq \frac{q^{K(K-1)/2} (|a|q^n)^K}{(q; q)_K} \exp(1/2) < \frac{2q^{K(K-1)/2} (|a|q^n)^K}{(q; q)_K}. \end{aligned}$$

□

The Jacobi theta functions are defined as

$$(17) \quad \theta_1(z; q) := \theta_1(v|\tau) := -i \sum_{k=-\infty}^{\infty} (-1)^k q^{(k+1/2)^2} e^{(2k+1)\pi i v},$$

$$(18) \quad \theta_2(z; q) := \theta_2(v|\tau) := \sum_{k=-\infty}^{\infty} q^{(k+1/2)^2} e^{(2k+1)\pi i v},$$

$$(19) \quad \theta_3(z; q) := \theta_3(v|\tau) := \sum_{k=-\infty}^{\infty} q^{k^2} e^{2k\pi i v},$$

$$(20) \quad \theta_4(z; q) := \theta_4(v|\tau) := \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} e^{2k\pi i v},$$

where

$$(21) \quad z = e^{2\pi i v}, \quad q = e^{\pi i \tau}, \quad \Im(\tau) > 0.$$

The Jacobi's triple product identities are

$$(22) \quad \theta_1(v|\tau) = 2q^{1/4} \sin \pi v (q^2; q^2)_\infty (q^2 e^{2\pi i v}; q^2)_\infty (q^2 e^{-2\pi i v}; q^2)_\infty,$$

$$(23) \quad \theta_2(v|\tau) = 2q^{1/4} \cos \pi v (q^2; q^2)_\infty (-q^2 e^{2\pi i v}; q^2)_\infty (-q^2 e^{-2\pi i v}; q^2)_\infty,$$

$$(24) \quad \theta_3(v|\tau) = (q^2; q^2)_\infty (-q e^{2\pi i v}; q^2)_\infty (-q e^{-2\pi i v}; q^2)_\infty,$$

$$(25) \quad \theta_4(v|\tau) = (q^2; q^2)_\infty (q e^{2\pi i v}; q^2)_\infty (q e^{-2\pi i v}; q^2)_\infty,$$

they satisfy transformations:

$$(26) \quad \theta_1\left(\frac{v}{\tau} \mid -\frac{1}{\tau}\right) = -i\sqrt{\frac{\tau}{i}}e^{\pi iv^2/\tau}\theta_1(v \mid \tau),$$

$$(27) \quad \theta_2\left(\frac{v}{\tau} \mid -\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}e^{\pi iv^2/\tau}\theta_4(v \mid \tau),$$

$$(28) \quad \theta_3\left(\frac{v}{\tau} \mid -\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}e^{\pi iv^2/\tau}\theta_3(v \mid \tau),$$

$$(29) \quad \theta_4\left(\frac{v}{\tau} \mid -\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}e^{\pi iv^2/\tau}\theta_2(v \mid \tau).$$

The Dedekind $\eta(\tau)$ is defined as [5]

$$(30) \quad \eta(\tau) := e^{\pi i\tau/12} \prod_{k=1}^{\infty} (1 - e^{2\pi i k \tau}),$$

or

$$(31) \quad \eta(\tau) = q^{1/12}(q^2; q^2)_{\infty}, \quad q = e^{\pi i\tau}, \quad \Im(\tau) > 0,$$

it has the transformation formula

$$(32) \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\eta(\tau).$$

Lemma 1.2. *For*

$$(33) \quad 0 < a < 1, \quad n \in \mathbb{N}, \quad \gamma > 0,$$

and

$$(34) \quad q = e^{-2\pi\gamma^{-1}n^{-a}},$$

we have

$$(35) \quad (q; q)_{\infty} = \sqrt{\gamma n^a} \exp\left\{\frac{\pi}{12}((\gamma n^a)^{-1} - \gamma n^a)\right\} \left\{1 + \mathcal{O}\left(e^{-2\pi\gamma n^a}\right)\right\},$$

and

$$(36) \quad \frac{1}{(q; q)_{\infty}} = \frac{\exp\left\{\frac{\pi}{12}(\gamma n^a - (\gamma n^a)^{-1})\right\}}{\sqrt{\gamma n^a}} \left\{1 + \mathcal{O}\left(e^{-2\pi\gamma n^a}\right)\right\}$$

as $n \rightarrow \infty$.

Proof. From formulas (30), (31) and (32) we get

$$\begin{aligned} (q; q)_{\infty} &= \exp(\pi\gamma^{-1}n^{-a}/12) \eta(\gamma^{-1}n^{-a}i) \\ &= \sqrt{\gamma n^a} \exp(\pi\gamma^{-1}n^{-a}/12) \eta(\gamma n^a i) \\ &= \sqrt{\gamma n^a} \exp(\pi\gamma^{-1}n^{-a}/12 - \pi\gamma n^a/12) \prod_{k=1}^{\infty} (1 - e^{-2\pi\gamma k n^a}) \\ &= \sqrt{\gamma n^a} \exp(\pi\gamma^{-1}n^{-a}/12 - \pi\gamma n^a/12) \left\{1 + \mathcal{O}\left(e^{-2\pi\gamma n^a}\right)\right\}, \end{aligned}$$

and

$$\frac{1}{(q; q)_{\infty}} = \frac{\exp(\pi\gamma n^a/12 - \pi\gamma^{-1}n^{-a}/12)}{\sqrt{\gamma n^a}} \left\{1 + \mathcal{O}\left(e^{-2\pi\gamma n^a}\right)\right\}$$

as $n \rightarrow \infty$. □

2. MAIN RESULTS

For $\Re(z) > -\frac{1}{2}$, we write

$$(37) \quad \frac{\Gamma_q(z + 1/2)}{(q; q)_\infty (1 - q)^{1/2 - z}} = \frac{1}{(q^{z+1/2}; q)_\infty},$$

then,

$$(38) \quad \Gamma_q(z + \frac{1}{2}) = \frac{(q; q)_\infty}{(1 - q)^{z-1/2}} \sum_{k=0}^{\infty} \frac{q^{k(z+1/2)}}{(q; q)_k}.$$

Formula (37) implies

$$(39) \quad \Gamma_q(\frac{1}{2} + z) \Gamma_q(\frac{1}{2} - z) = \frac{(1 - q)(q; q)_\infty^3}{(q, q^{1/2-z}, q^{1/2+z}; q)_\infty},$$

or

$$(40) \quad \Gamma_q(\frac{1}{2} + z) \Gamma_q(\frac{1}{2} - z) = \frac{(1 - q)(q; q)_\infty^3}{\theta_4(q^z; q^{1/2})}.$$

Thus,

$$(41) \quad \Gamma_q(\frac{1}{2} - z) = \frac{(q; q)_\infty^2 (1 - q)^{z+1/2}}{\theta_4(q^z; q^{1/2})} (q^{z+1/2}; q)_\infty$$

or

$$(42) \quad \Gamma_q(\frac{1}{2} - z) = \frac{(q; q)_\infty^2 (1 - q)^{z+1/2}}{\theta_4(q^z; q^{1/2})} \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} (-q^{z+1/2})^k}{(q; q)_k}$$

for $\Re(z) > -\frac{1}{2}$.

2.1. Case $q \rightarrow 1$ and $z \rightarrow \infty$:

Theorem 2.1. *For*

$$(43) \quad 0 < a < \frac{1}{2}, \quad n \in \mathbb{N}, \quad u \in \mathbb{R}, \quad q = \exp(-2n^{-a}\pi),$$

we have

$$(44) \quad \frac{1}{\Gamma_q(\frac{1}{2} - n - n^a u)} = \frac{2 \exp(\pi n^{-a}(n^a u + n)^2) \cos \pi(n^a u + n) \{1 + \mathcal{O}(e^{-2\pi n^a})\}}{\sqrt{n^a} \exp(\pi n^a/12 + \pi n^{-a}/6) (1 - \exp(-2\pi n^{-a}))^{n+n^a u+1/2}},$$

and

$$(45) \quad \frac{1}{\Gamma_q(\frac{1}{2} + n + n^a u)} = \frac{\exp(\pi n^a/12 - \pi n^{-a}/12) \{1 + \mathcal{O}(e^{-2\pi n^a})\}}{\sqrt{n^a} (1 - e^{-2\pi n^{-a}})^{1/2 - n - n^a u}}$$

as $n \rightarrow \infty$, and the big- \mathcal{O} term is uniform with respect u for $u \in [0, \infty)$.

2.2. Case $q \rightarrow 1$ and z fixed:

Theorem 2.2. *Assume that*

$$(46) \quad q = e^{-2\pi\tau}, \quad \tau > 0, \quad x \in \mathbb{R}.$$

If

$$(47) \quad x > -\frac{1}{2}, \quad q > 1 - \exp(-2^{2x+1}),$$

then

$$(48) \quad \Gamma_q\left(\frac{1}{2} + x\right) = \Gamma(x + 1/2) \left\{1 + \mathcal{O}\left((1-q) \log^2(1-q)\right)\right\},$$

and

$$(49) \quad \frac{1}{\Gamma_q\left(\frac{1}{2} - x\right)} = \frac{\left\{1 + \mathcal{O}\left((1-q) \log^2(1-q)\right)\right\}}{\Gamma\left(\frac{1}{2} - x\right)},$$

where the implicit constants of the big-O terms are independent of x under the condition (47)

3. PROOFS

3.1. Proof for Theorem 2.1.

Proof. We first observe that

$$\frac{1}{(q; q)_\infty} = \frac{\exp(\pi n^a/12 - \pi n^{-a}/12)}{\sqrt{n^a}} \left\{1 + \mathcal{O}\left(e^{-2\pi n^a}\right)\right\},$$

and

$$\begin{aligned} \frac{1}{\Gamma_q(1/2 - n - n^a u)} &= \frac{(q^{1/2-n} e^{2\pi u}; q)_\infty}{(q; q)_\infty (1-q)^{n+n^a u+1/2}} \\ &= \frac{(q, q^{1/2} e^{-2\pi u}, q^{1/2} e^{2\pi u}; q)_\infty q^{-n^2/2} e^{2\pi n u}}{(-1)^n (1-q)^{n+n^a u+1/2} (q, q, q^{n+1/2} e^{-2\pi u}; q)_\infty} \end{aligned}$$

as $n \rightarrow \infty$. Then we have

$$\frac{1}{(q, q, q^{n+1/2} e^{-2\pi u}; q)_\infty} = n^{-a} \exp(\pi n^a/6 - \pi n^{-a}/6) \left\{1 + \mathcal{O}\left(e^{-2\pi n^a}\right)\right\},$$

and

$$\begin{aligned} (q, q^{1/2} e^{-2\pi u}, q^{1/2} e^{2\pi u}; q)_\infty &= \theta_4(ui \mid n^{-a}i) = n^{a/2} e^{\pi n^a u^2} \theta_2(n^a u \mid n^a i) \\ &= 2n^{a/2} \exp \pi n^a (u^2 - 1/4) \cos(n^a u \pi) \left\{1 + \mathcal{O}\left(e^{-2\pi n^a}\right)\right\} \end{aligned}$$

as $n \rightarrow \infty$. Thus,

$$\frac{1}{\Gamma_q(1/2 - n - n^a u)} = \frac{2 \exp \pi (n^{-a} (n^a u + n)^2) \cos(\pi n^a u + n\pi) \left\{1 + \mathcal{O}\left(e^{-2\pi n^a}\right)\right\}}{n^{a/2} (1 - e^{-2\pi n^a})^{n+n^a u+1/2} \exp(\pi n^a/12 + \pi n^{-a}/6)}$$

as $n \rightarrow \infty$, and it is clear that the big-O term is uniform with respect to $u \geq 0$.

Similarly, formula (45) follows from Lemma 1.1 and 1.2. \square

3.2. Proof for Theorem 2.2.

Proof. From (27), (32) and

$$\Gamma_q\left(\frac{1}{2} + x\right)\Gamma_q\left(\frac{1}{2} - x\right) = \frac{\eta(\tau i)^3 e^{\pi\tau/4} (1 - e^{-2\pi\tau})}{\theta_4(x\tau i | \tau i)},$$

we get

$$\Gamma_q\left(\frac{1}{2} + x\right) = \frac{(1 - e^{-2\pi\tau}) \exp(-\pi\tau(x^2 - 1/4))}{\tau \Gamma_q(\frac{1}{2} - x)} \frac{\eta^3(i/\tau)}{\theta_2(x|i/\tau)},$$

and (8) and (15) imply that

$$\begin{aligned} \Gamma_q\left(\frac{1}{2} + x\right) &= \frac{(1 - e^{-2\pi\tau}) \exp(-\pi\tau(x^2 - 1/4))}{2\tau \cos(\pi x) \Gamma_q(\frac{1}{2} - x)} \left\{1 + \mathcal{O}\left(\exp\left(-\frac{2\pi}{\tau}\right)\right)\right\} \\ &= \frac{\pi \exp(-\pi\tau x^2) \{1 + \mathcal{O}(\tau)\}}{\cos(\pi x) \Gamma_q(\frac{1}{2} - x)} \\ &= \frac{\pi \exp\left(-\pi\tau \log^2(1 - q) \frac{x^2}{\log^2(1 - q)}\right) \{1 + \mathcal{O}(\tau)\}}{\cos(\pi x) \Gamma_q(\frac{1}{2} - x)} \end{aligned}$$

as $\tau \rightarrow 0^+$ and the big-O term is independent of x .

The condition (47) implies that

$$0 < \frac{x^2}{\log^2(1 - q)} < 1,$$

then,

$$\Gamma_q\left(\frac{1}{2} + x\right) = \frac{\pi \{1 + \mathcal{O}((1 - q) \log^2(1 - q))\}}{\cos(\pi x) \Gamma_q(\frac{1}{2} - x)}$$

as $q \rightarrow 1$ and the implicit constant above is independent of x .

It is well-known that an q -analogue of (6) is [1, 2, 3, 4]

$$\int_0^\infty \frac{t^{x-1/2}}{(-t; q)_\infty} dt = \frac{\pi}{\cos \pi x} \frac{(q^{1/2-x}; q)_\infty}{(q; q)_\infty}, \quad x > -\frac{1}{2},$$

or

$$\int_0^\infty \frac{t^{x-1/2} dt}{(-(1 - q)t; q)_\infty} = \frac{\pi}{\cos(\pi x) \Gamma_q(\frac{1}{2} - x)}, \quad x > -\frac{1}{2}.$$

Consequently,

$$\Gamma_q\left(\frac{1}{2} + x\right) = \{1 + \mathcal{O}((1 - q) \log^2(1 - q))\} \int_0^\infty \frac{t^{x-1/2} dt}{(-(1 - q)t; q)_\infty}$$

as $\tau \rightarrow 0^+$ and the implicit constant of the big-O term here is independent of x with $x > -\frac{1}{2}$.

Write

$$\int_0^\infty \frac{t^{x-1/2} dt}{(-(1 - q)t; q)_\infty} := I_1 + I_2,$$

where

$$I_1 := \int_0^{\log(1-q)^{-2}} \frac{t^{x-1/2} dt}{(-(1 - q)t; q)_\infty},$$

and

$$I_2 := \int_{\log(1-q)^{-2}}^\infty \frac{t^{x-1/2} dt}{(-(1 - q)t; q)_\infty}.$$

In I_1 we have

$$\begin{aligned}
 \log(-(1-q)t; q)_\infty &= \sum_{k=0}^{\infty} \log(1 + (1-q)tq^k) \\
 &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} ((1-q)tq^k)^{n+1} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1} \frac{(1-q)^{n+1}}{1-q^{n+1}} \\
 &= t + r(n),
 \end{aligned}$$

where

$$r(n) := \sum_{n=1}^{\infty} \frac{(-1)^n t^{n+1}}{n+1} \frac{(1-q)^{n+1}}{1-q^{n+1}}.$$

The condition (47) implies that

$$0 < (1-q) \log(1-q)^{-1} < e^{-1},$$

then

$$|r(n)| \leq c(1-q) \log^2(1-q),$$

with

$$c = 4 \sum_{n=0}^{\infty} \frac{[2(1-q) \log(1-q)^{-1}]^n}{n+2} < 4 \sum_{n=0}^{\infty} \frac{(2e^{-1})^n}{n+2}.$$

Therefore,

$$I_1 = \int_0^{2 \log(1-q)^{-1}} e^{-t} t^{x-1/2} dt \{1 + \mathcal{O}((1-q) \log^2(1-q))\}$$

as $\tau \rightarrow 0^+$, and the implicit constant of the big-O term is independent of x .

Since

$$\int_{2 \log(1-q)^{-1}}^{\infty} e^{-t} t^{x-1/2} dt \leq e^{-\log(1-q)^{-1}} \int_{2 \log(1-q)^{-1}}^{\infty} e^{-t/2} t^{x-1/2} dt < (1-q) \Gamma(x+1/2) 2^{x+1/2},$$

clearly,

$$\int_{2 \log(1-q)^{-1}}^{\infty} e^{-t} t^{x-1/2} dt = \Gamma(x+1/2) \mathcal{O}((1-q) \log^2(1-q)).$$

Thus,

$$I_1 = \Gamma(x+1/2) \{1 + \mathcal{O}((1-q) \log^2(1-q))\},$$

as $\tau \rightarrow 0^+$, and the implicit constant of the big-O term is independent of x under the condition (47).

Recall that

$$(-(1-q)t; q)_\infty = \sum_{n=0}^{\infty} q^{n(n-1)/2} t^n \frac{(1-q)^n}{(q; q)_n} > q^{n(n-1)/2} t^n \frac{(1-q)^n}{(q; q)_n},$$

for any $n = \lfloor -\log(1-q) \rfloor \geq 2^{2x+1} > 2x+1$. Then,

$$\begin{aligned} I_2 &\leq \frac{q^{n(1-n)/2}(q; q)_n (-\log(1-q))^{2x-2n+1}}{(1-q)^n(n-x-1/2)} \\ &< \frac{\Gamma(x+1/2)n!q^{n(1-n)/2}(-\log(1-q))^{2x-2n+1}}{\Gamma(x+3/2)} \\ &< \Gamma(x+1/2)n!q^{n(1-n)/2}(-\log(1-q))^{2x-2n+1}. \end{aligned}$$

It is clear that

$$q^{n(1-n)/2} = \mathcal{O}(1)$$

as $\tau \rightarrow 0^+$. From the Stirling formula

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left\{1 + \mathcal{O}\left(\frac{1}{n}\right)\right\}$$

as $n \rightarrow \infty$, we have

$$I_2 = \Gamma(x + \frac{1}{2})\mathcal{O}\left((1-q)\log^{1/2}(1-q)^{-1}\right)$$

as $\tau \rightarrow 0^+$ and the implicit constant of the big-O term is independent of x under the condition (47).

Therefore, under the condition (47)

$$\int_0^\infty \frac{t^{x-1/2}dt}{(-(1-q)t; q)_\infty} = \Gamma(x+1/2) \{1 + \mathcal{O}((1-q)\log^2(1-q))\}$$

as $\tau \rightarrow 0^+$ and the implicit constant of the big-O term is independent of $x > -\frac{1}{2}$.

Hence we have proved that under the condition (47) we have

$$\Gamma_q\left(\frac{1}{2} + x\right) = \Gamma(x + \frac{1}{2}) \{1 + \mathcal{O}((1-q)\log^2(1-q))\}, \quad x > -1/2.$$

Then,

$$\Gamma_q\left(\frac{1}{2} - x\right) = \frac{\pi \sec \pi x}{\Gamma(x+1/2)} \{1 + \mathcal{O}((1-q)\log^2(1-q))\},$$

which is

$$\frac{1}{\Gamma_q(\frac{1}{2} - x)} = \frac{\{1 + \mathcal{O}((1-q)\log^2(1-q))\}}{\Gamma(\frac{1}{2} - x)},$$

where the implicit constant of the big-O term is independent of $x > -\frac{1}{2}$. \square

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